Hyperbolic disc

venerdì 7 ottobre 2022 14:07

DISC MODEL

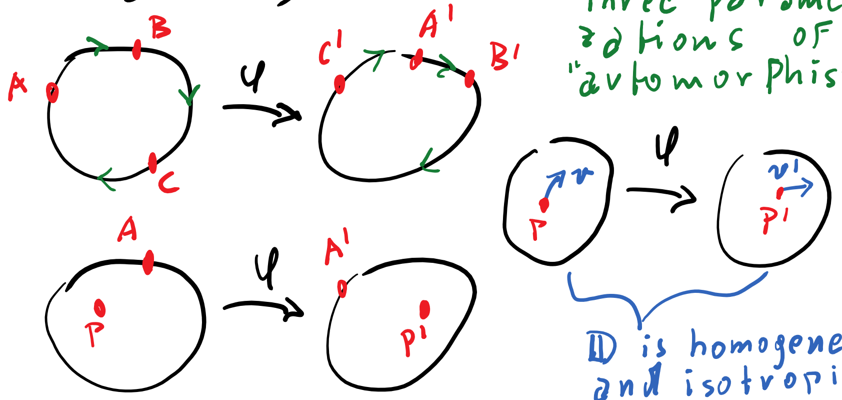
$\mathbb{D} = \{ z : |z| < 1 \} \subseteq \mathbb{C}$

$\mathbb{T} = \partial \mathbb{D} = \{ e^{is} : s \in \mathbb{R} \}$

$$\varphi(z) = e^{is} \cdot \frac{a - z}{1 - \bar{a} z} \qquad s \in \mathbb{R}, \ |a| < 1$$

$\mathbb{D} \xrightarrow{\varphi} \mathbb{D}$: bijection

Three parametrizations of such "automorphisms"



$\mathbb{D}$ is homogeneous and isotropic w.r.t. automorphisms

$$|\varphi(e^{it})| = \left| \frac{a - e^{it}}{1 - \bar{a} e^{it}} \right| = 1$$

$$\boxed{\begin{array}{l} \text{Schwarz Lemma} \quad \mathbb{D} \xrightarrow{f} \mathbb{D} \ \text{holomorphic}, \ f(0) = 0 \\ \Rightarrow \ |f(z)| \le |z| \qquad \exists z \ne 0 \ (=) \Leftrightarrow f(z) = \lambda z \quad |\lambda| = 1 \\ \quad\ \ |f'(0)| \le 1 \qquad (=) \Leftrightarrow \text{same} \end{array}}$$

MAGIC RELATIONS

$$|\varphi'(z)| = \frac{|1 - (1 - \bar{a} z) - (a - z)(-\bar{a})|}{|1 - \bar{a} z|^2} = \frac{1 - |a|^2}{|1 - \bar{a} z|^2}$$

$$1 - |\varphi(z)|^2 = \frac{|1 - \bar{a} z|^2 - |a - z|^2}{|1 - \bar{a} z|^2} = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \bar{a} z|^2}$$

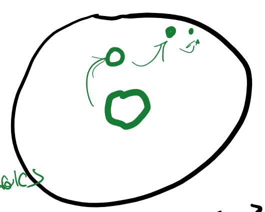
Hyperbolic Metric (Riemann, Beltrami, Poincaré)

$$ds^2 = \frac{4 \cdot |dz|^2}{(1 - |z|^2)^2}$$

$d(z, z + dz)$

$\wr\wr$

$\dfrac{2 \cdot |dz|}{1 - |z|^2}$



MAGIC

$$\frac{|d\varphi(z)|}{\cdots} = \frac{|\varphi'(z)| \cdot |dz|}{\cdots} \overset{*}{=} \frac{1 - |a|^2}{\cdots} \cdot \frac{|1 - \bar{a} z|^2 |dz|}{\cdots}$$

MAGIC

$$\frac{|d\varphi(z)|}{1-|\varphi(z)|^2} = \frac{|\varphi'(z)|\cdot|dz|}{1-|\varphi(z)|^2} = \frac{\cancel{1-|a|^2}}{\cancel{|1-\bar a z|^2}}\cdot\frac{\cancel{|1-\bar a z|^2}\,|dz|}{(\cancel{1-|a|^2})|1-|z|^2|}$$

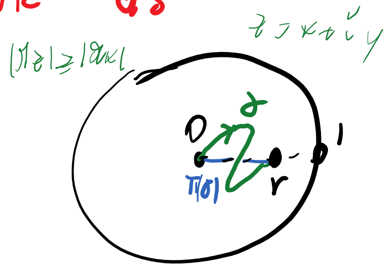
$$= \frac{|dz|}{1-|z|^2}:$$ ALL φ's (AND JUST THEM) ARE (SENSE PRESERVING) ISOMETRIES FOR ds^2

$0<r<1$

$$\text{LENGTH}(\gamma) := 2\int_\gamma \frac{|dz|}{1-|z|^2}$$

$$\geq 2\cdot\int_{\pi(\gamma)}\frac{|dx|}{1-x^2} \geq 2\cdot\int_0^r \frac{dx}{1-x^2}$$

$$= \int_0^r\left(\frac{1}{1-x}+\frac{1}{1-x}\right)dx = \log\frac{1+r}{1-r} = \text{LENGTH}\,[0,r]$$

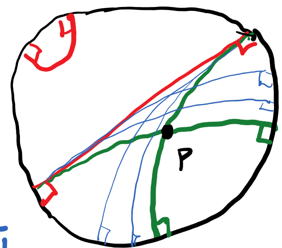


$$\boxed{d(0,r) = \log\frac{1+r}{1-r}}$$ Hyperbolic (non Euclidean) distance

Moving things by φ's:

(I) GEODESICS ARE CIRCLES/LINES $\perp$ TO $\mathbb{T}$

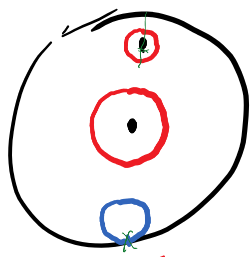
(AND THE AXIOM OF PARALLELS FAILS)



(II) $$d(z,w) = \log\frac{1+\left|\frac{z-w}{1-\bar z w}\right|}{1-\left|\frac{z-w}{1-\bar z w}\right|}$$

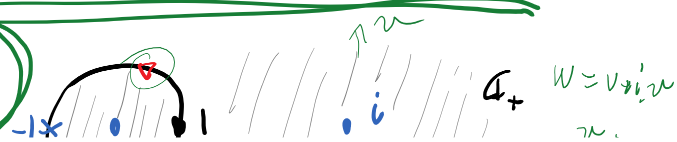
(III) DISCS IN THE METRIC ARE EUCLIDEAN DISCS:

BUT CENTERS ARE DIFFERENT

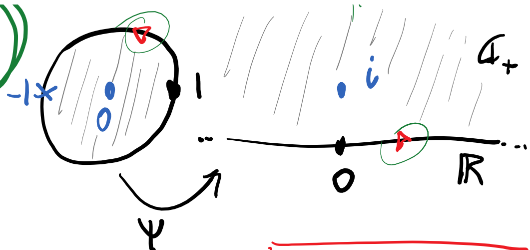


A SPECIAL ROLE IS THAT OF (EUCLIDEAN) CIRCLES TANGENT TO $\mathbb{T}$ (HOROCYCLES).

UPPER HALF PLANE MODEL



UPPER HALF PLANE MODEL



$\mathbb{C}_+$ $\quad w = v + i\nu$, $\nu > 0$

$\Psi(-1) = \infty$

$\Psi(1) = 0$

$\Psi(0) = i$

$$z \longmapsto \boxed{w = i \cdot \frac{1-z}{1+z}}$$

$$\Psi(e^{it}) = i\,\frac{1-e^{it}}{1+e^{it}} = i\,\frac{(-2i)\cdot\sin(t/2)}{2\cdot\cos(t/2)} = tg(t/2)$$

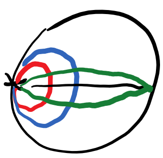
$$ds := \frac{2|dz|}{1-|z|^2}$$

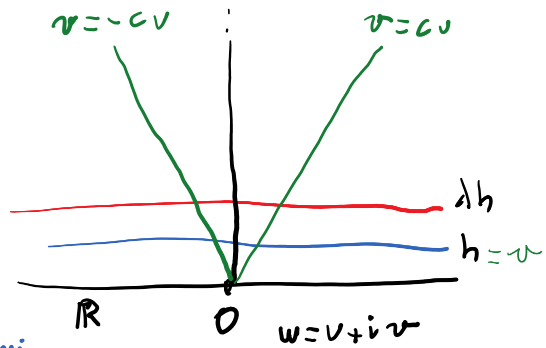
$$\boxed{z = \frac{i-w}{i+w} \qquad \frac{dz}{dw} = \frac{-2i}{(i+w)^2}}$$

$$= \frac{|dw|}{\operatorname{Im}(w)} : \text{ hyperbolic metric in } \mathbb{C}_+$$

$\nu = \operatorname{Im}(w)$

$\nu = -cv \qquad v = cv$





Observe!

The hyperbolic metric restricted to the horocycle $\boxed{\nu = h}$ is $ds = \frac{dv}{h}$:

a rescaling of the Euclidean metric on $\mathbb{R}$

Exercise:

$$\begin{array}{ccccc} \mathbb{D} & z & \longmapsto & w = i\,\dfrac{1-z}{1+z} & \mathbb{C}_+ \\ & \downarrow & & \downarrow & \boxed{\lambda > 0} \\ \mathbb{D} & \dfrac{z-r}{1-rz} = \zeta & \longmapsto & \xi = i\,\dfrac{1-\zeta}{1+\zeta} = \lambda \cdot w & \mathbb{C}_+ \end{array}$$

$$\Longleftrightarrow \quad \lambda = \frac{1+r}{1-r}$$

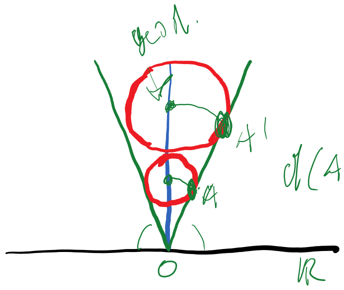
AUTOMORPHISMS OF $\mathbb{C}_+$

$$\xi = \frac{aw+b}{cw+d} \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \quad a,b,c,d \in \mathbb{R}, \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1$$

$w \longrightarrow w + a$

- $\mathbb{R}$ - translations $\quad w \mapsto w + a, \ a \in \mathbb{R}$ $\qquad \mathbb{R}$
- dilations $\quad w \mapsto \lambda w, \ \lambda \in (0, \infty)$

- dilations $\quad w \mapsto \lambda w, \ \lambda \in (0,\infty)$
- inversion $\quad w \mapsto -\frac{1}{w} \quad \mathbb{Z}_2$



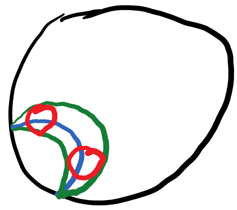
The discs are isometric (dilation)

$d(A, \text{blue}) = d(A', \text{blue})$

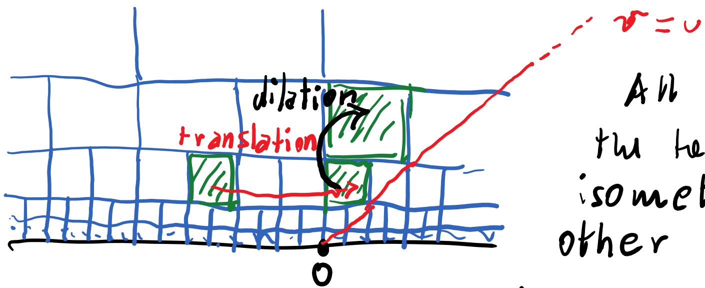
$\Downarrow$

CIRCLES MEETING $\partial\mathbb{D}$ AT ANGLES $\neq 0$ ARE LOCI OF POINTS HAVING FIXED DISTANCE FROM A GEODESIC

THE DISTANCE DEPENDS ON THE ANGLE.



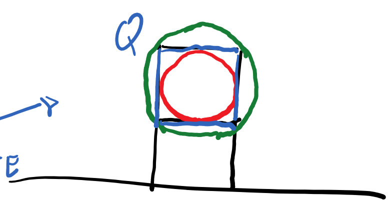
DYADIC PICTURES:



All squares in the tessellation are isometric to each other by either translations or dilations:

$\mathbb{C}_+$ has a tessellation by isometric copies of the same tile:

EACH TILE IS AN APPROXIMATE HYPERBOLIC DISC:



$$\exists\, c < G : D(w(Q), c) \subseteq Q \subseteq D(\zeta(Q), G)$$

hyperbolic disc

center

hyperbolic radius

We can obtain the same containment

with $w(Q)=\zeta(Q)$ by choosing suitable c, Q.

The Pseudo-hyperbolic distance and its Functional analytic meaning

$z, w \in \mathbb{D}$:

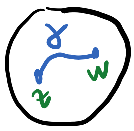
$$\delta(z,w) := \left|\frac{z-w}{1-\bar{z}w}\right| \in [0,1)$$

- It is a distance

$$d = \log\frac{1+\delta}{1-\delta} \quad ; \quad \delta = \tanh(d/2) = \delta(d)$$

$d \mapsto \delta(d)$ is strictly increasing, concave and $\delta(0)=0$; d is a distance $\Rightarrow$ δ is a distance

- $\delta(z, z+dz) = \dfrac{|dz|}{1-|z|^2} = d(z, z+dz)$, hence, the length distance associated with δ is d



$[a,b] \xrightarrow{\gamma} \mathbb{D}$, $\gamma(a)=z$, $\gamma(b)=w$ $\quad\Big|\quad$ $\Lambda(\gamma) := \sup\left\{\sum_{j=1}^{n} \delta(\gamma(t_j), \gamma(t_{j-1})) : a = t_0 < t_1 < \dots < t_n = b\right\} = \text{length}(\gamma)$

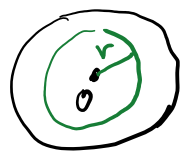
hence, $\inf\{\Lambda(\gamma) : \gamma \text{ s.t. } \gamma(a)=z, \gamma(b)=w\} = d(z,w)$

Hardy space $H^2(\mathbb{D}) \ni f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$:

$$\|f\|^2_{H^2} := \sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty$$

$$= \sup_{0<r<1} \ (\Vert\ \lim_{r\to 1}) \ \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{it})|^2 \, dt$$

- Hilbert space
- f converges for $|z|<1$



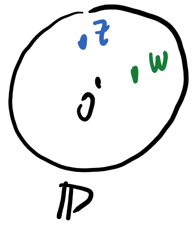
Calculations:

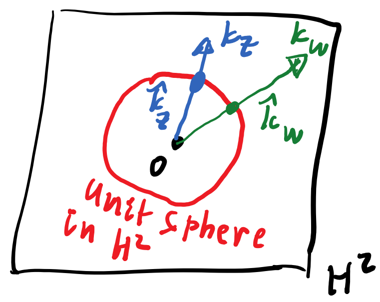
$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n = \sum_{n=0}^{\infty} \hat{f}(n) \overline{(\bar{z}^n)} = \left\langle \sum_{n=0}^{\infty} \hat{f}(n) w^n, \sum_{n=0}^{\infty} \bar{z}^n w^n \right\rangle_{H^2}$$

$$= \langle f, k_z \rangle_{H^2} \quad \text{where} \quad k_z(w) = \sum_{n=0}^{\infty} \bar{z}^n w^n = \frac{1}{1-\bar{z}w}$$

- $k_z(w) = \langle k_z, k_w \rangle_{H^2} = \overline{\langle k_w, k_z \rangle} = \overline{k_w(z)}$
- $\|k_z\|^2_{H^2} = \langle k_z, k_z \rangle_{H^2} = k_z(z) = \dfrac{1}{1-|z|^2}$

$\{k_z\}_{z\in\mathbb{D}}$ IS THE REPRODUCING KERNEL OF $H^2(\mathbb{D})$



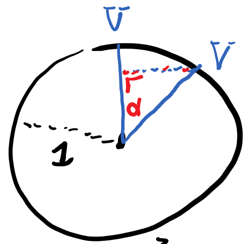


$$\hat{k}_z = \frac{k_z}{\|k_z\|_{H^2}} = \sqrt{1-|z|^2}\cdot k_z ;$$

PROJECTION OF k_z ONTO THE UNIT SPHERE OF H^2.

GLEASON DISTANCE BETWEEN $z, w \in \mathbb{D}$:

$$D(z,w) := \sqrt{1-|\langle \hat{k}_z, \hat{k}_w \rangle|^2} \overset{*}{=} \delta(z,w) \text{ !!!}$$



$$D(v,V) = \sqrt{1-\cos^2\alpha} = \sqrt{2}\cdot|\sin\alpha| ;$$

if v, V are real vectors

$$* \quad D(z,w)^2 = 1 - \frac{(1-|z|^2)(1-|w|^2)}{|1-\bar{z}w|^2} \underset{\uparrow}{=} \left|\frac{z-w}{1-\bar{z}w}\right|^2$$

by magics

Hardy on $\mathbb{C}^+$

$$H^2(\mathbb{C}_+) \ni f(z) = \frac{1}{2\pi}\int_0^{+\infty} \hat{f}(\xi)\, e^{iz\xi}\, d\xi : \quad \left(\text{Holomorphic on } \mathbb{C}_+\right)$$

$$\infty > \|f\|^2_{H^2(\mathbb{C}_+)} := \frac{1}{2\pi}\int_0^{+\infty} |\hat{f}(\xi)|^2\, d\xi \qquad |e^{iz\xi}| = e^{-y\xi}$$

$$= \sup_{h>0} \text{ or } \lim_{h\to 0} \int_{-\infty}^{+\infty} |f(x+ih)|^2\, dx$$

they've the same provided $\sup_h(\cdots) < \infty$

Reproducing kernel:

$$f(z) = \langle f, h_z \rangle_{H^2(\mathbb{C})} \quad \text{with}$$

$$\hat{h}_z(\xi) = e^{-i\bar{z}\xi} \quad \text{i.e.}$$

$$h_z(w) = \frac{1}{2\pi}\int_0^{+\infty} e^{-i\bar{z}\xi}\, e^{i\xi w}\, d\xi$$

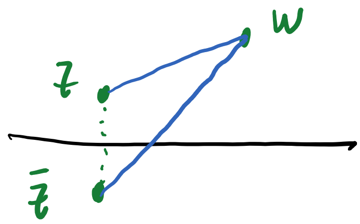
$$h_z(w) = \frac{1}{2\pi}\int_0^{\infty} e^{-is\bar z} e^{isw}\,ds$$

$$= \frac{1}{2\pi}\left[\frac{e^{is(w-\bar z)}}{i\,(w-\bar z)}\right]_0^{\infty} = \frac{i}{2\pi}\,\frac{1}{w-\bar z}$$

$$\bullet \quad \|h_z\|^2_{H^2} = \frac{i}{2\pi}\,\frac{1}{z-\bar z} = \frac{1}{4\pi y} \qquad \begin{matrix} z = x+iy \\ w = v+iv \end{matrix}$$

GLEASON DISTANCE: $D(z,w)^2 = 1 - \dfrac{4vy}{|w-\bar z|^2}$

$$= \left|\frac{w-z}{w-\bar z}\right|^2 \overset{=}{\underset{\text{DIRECT CALCULATION}}{}} \delta_{\mathbb{C}_+}(z,w)^2 := \delta_{\mathbb{D}}\left(\psi^{-1}(z), \psi^{-1}(w)\right)^2$$



Curvature For a conformal Riemannian metric $ds^2 = e^{\sigma(z)}|dz|^2$

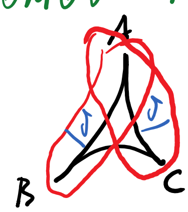
the Gauss curvature is:

$$k = -\frac{\Delta\sigma}{2\cdot e^{2\sigma}}$$

For $ds^2 = \dfrac{4|dz|^2}{(1-|z|^2)^2} = e^{\log 4 - 2\log(1-|z|^2)}|dz|^2$

we have $\boxed{k = -1}$ constant negative curvature

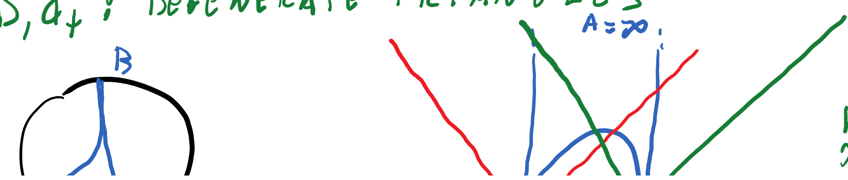
GROMOV HYPERBOLICITY: "THIN TRIANGLES" IN METRIC SPACES WITH GEODESICS



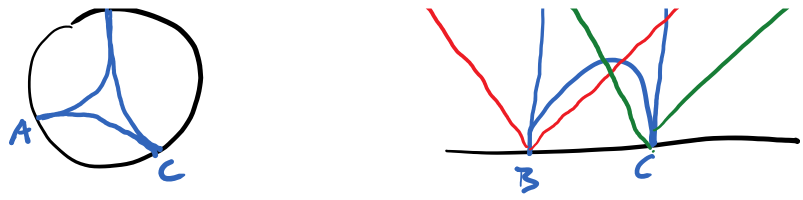
$\exists\, \delta > 0 : \forall\, A, B, C$

δ-FATTENING(AB) ∪ δ-FATTENING(AC) ⊇ BC

$\mathbb{D}, \mathbb{C}_+$: DEGENERATE TRIANGLES

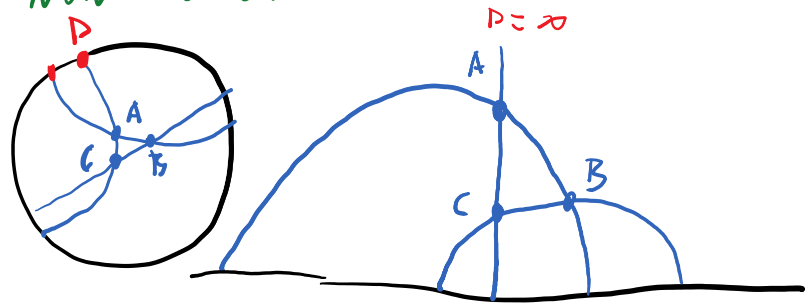


Recall FATTENING OF GEODESICS

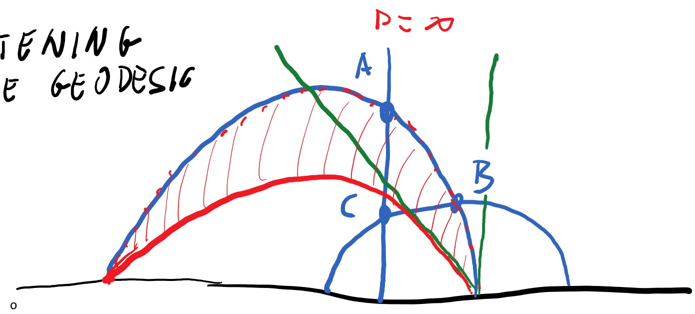


RECALL
FATTENING
OF GEODESICS
FOR ds^2

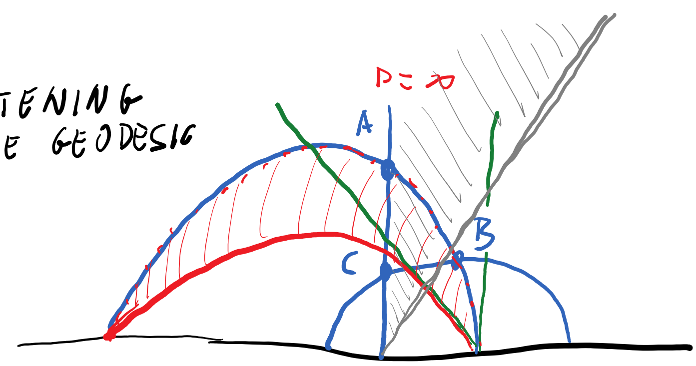
NON DEGENERATE TRIANGLES



FATTENING
OF THE GEODESIC
AB



AND FATTENING
OF THE GEODESIC
AC



CB IS CONTAINED IN THE UNION OF THE TWO FATTENINGS, ALTHOUGH HERE WE FATTENED TWO INFINITE GEODESICS

CONTRACTIVE PROPERTIES OF HOLOMORPHIC MAPS $\mathbb{D} \xrightarrow{f} \mathbb{D}$

$\mathbb{D} \xrightarrow{f} \mathbb{D}$ HOLOMORPHIC

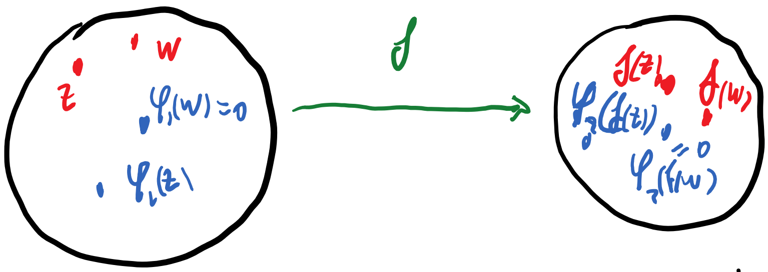
$\Downarrow$

$$d(f(z), f(w)) \le d(z, w)$$

and (=) holds $\exists z \ne w \iff$ f is an automorphism

and (=) holds $\exists\, z \neq w \iff$ f is an automorphism

Proof. Move $f(z)$, z to 0 by automorphisms and use Schwarz Lemma.



$$d(f(z), f(w)) = d(\varphi_2(f(z)), 0) = \log \frac{1+|\varphi_2(f(z))|}{1-|\varphi_2(f(z))|}$$

$$\leq \log \frac{1+|\varphi_1(z)|}{1-|\varphi_1(z)|} \quad \left\{ \begin{array}{l} \text{because } \varphi_2 \circ f \circ \varphi_1^{-1} : \mathbb{D} \to \mathbb{D} \\ \text{and } (\varphi_2 \circ f \circ \varphi_1^{-1})(0) = 0, \\ \text{so Schwarz applies} \end{array} \right.$$

$$= d(\varphi_1(z), \varphi_1(w))$$

$$= d(z, w) \quad \blacksquare$$

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